

A lattice Boltzmann method based on generalized polynomials

Rodrigo C. V. Coelho,¹ Anderson Ilha,² and Mauro M. Doria^{3,1}

¹*Departamento de Física dos Sólidos, Universidade Federal do Rio de Janeiro, 21941-972 Rio de Janeiro, Brazil*

²*Instituto Nacional de Metrologia, Normalização e Qualidade Industrial, Duque de Caxias 25.250-020 RJ Brazil*

³*Dipartimento di Fisica, Università di Camerino, I-62032 Camerino, Italy*

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We propose a lattice Boltzmann method based on the expansion of the equilibrium distribution function in powers of generalized orthonormal polynomials which are weighted by the equilibrium distribution function itself. The D-dimensional Euclidean space Hermite polynomials correspond to the particular weight of a gaussian function. The proposed polynomials give a general method to obtain an expansion of the equilibrium distribution function in powers of the ratio between the displacement velocity and the local scale velocity of the fluid.

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Introduction. – According to the Drude-Sommerfeld model the electrons in a metal form a gas under collisions similarly to the atoms in a rarefied gas whose time evolution Boltzmann described in his famous equation. Nevertheless quantum particles obey either the Fermi-Dirac (FD) or the Bose-Einstein (BE) statistics, and only for low densities and high temperatures become the well-known Maxwell-Boltzmann (MB) statistics satisfied by the atoms in a gas. A Boltzmann equation for particles obeying the BE-FD statistics was first considered by E. A. Uheling and G. E. Uhlenbeck¹. In many situations the scattering by collisions is sufficiently well described by the Bhatnagar-Gross-Krook (BGK) term^{2,3}. The Boltzmann-BGK equation has become the framework to understand the Drude-Sommerfeld model for the conduction of electrons in metals⁴. In the 1980's⁵ a revolutionary method was proposed to treat classical fluids using the Boltzmann-BGK equation that became known as the Lattice Boltzmann Method (LBM). It applies for low Mach number ($Ma \sim u/c_r$, u is the displacement velocity and $c_r \equiv \sqrt{kT_r/m}$ is essentially the speed of sound at temperature T_r). The LBM retrieves the continuity and the Navier-Stokes equations of hydrodynamics for low Knudsen number ($Kn \sim l/L$, l is the mean free path and L is the characteristic length scale of the system). However in many situations instead of the speed of sound there is another velocity in the fluid, c_0 , that sets the scale for the smallness of u . It is well known that room temperature electrons in metals are essentially governed by their zero temperature properties, and so the Fermi velocity v_F , and not c_r , is important. (for instance, for cooper⁴, $v_F = 1.57 \cdot 10^6$ m/s). The displacement velocity is very low in metals for typical electric fields, $u \sim 0.1$ - 1.0 m/s, that is, $u/v_F \sim 6.4$ - $64 \cdot 10^{-8}$, while Mach's number is not for $T_r = 0$, since $c_r = 0$. In this letter we obtain a LBM valid for the small ratio u/c_0 , that in case of electrons in metals corresponds to u/v_F instead of u/c_r . The Knudsen number remains small even in the treatment of microscale electronics since the collision time (cooper⁴) is $\tau = 2.1 \cdot 10^{-13}$ s which sets the mean free path as $l = v_F \tau \sim 0.33 \mu\text{m}$. To obtain this LBM we propose here a new set of D-dimensional orthonormal poly-

nomials, that generalize the Hermite polynomials. The D-dimensional Hermite polynomials have widespread use ranging from quantum optics⁶ to the Boltzmann equation⁷. Long ago, H. Grad^{7,8} pointed out the importance of expanding the distribution function in terms of them in the microscopic velocity space. The present LBM is applicable to classical thermal fluids and to semi-classical fluids⁹, such as electrons in metals, Bose-Einstein condensates, relativistic systems¹⁰ and graphene¹¹, without the need for the multiple decomposition in distinct sets of polynomials^{12,13}.

The time evolution of a gas in the Boltzmann-BGK

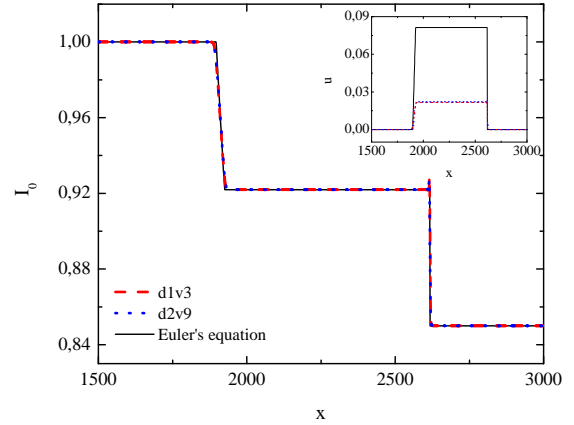


FIG. 1. Simulations are done in a one-dimensional system with $L_X \times L_Y = 3001 \times 1$ nodes with periodic boundary conditions. The initial velocity is zero everywhere and the density is 1.0 for $L_X/4 < x < 3L_X/4$ and 0.85 elsewhere. Only half of the system is shown since the other one is a mirror image. The relaxation time is $\tau = 0.58$.

equation is achieved through a statistical distribution function, constantly driven back to its equilibrium value, $f^{(eq)}(\xi, \mathbf{x})$, in a characteristic relaxation time τ . The *equilibrium distribution function* (EDF) depends on the microscopic velocity, ξ , and the position, \mathbf{x} and also de-

depends on local macroscopic parameters, such as the density $\rho(\mathbf{x})$ (or the chemical potential $\mu(\mathbf{x})$), the macroscopic (displacement) velocity $\mathbf{u}(\mathbf{x})$, and the temperature $\theta(\mathbf{x})$. For the quantum particles the Fermi-Dirac (FD,+) and the Bose-Einstein (BE,-) EDF's describe fermions and bosons, respectively. They correspond to $f_{FD/BE}^{(eq)} = 1/[\exp(-\mu/\theta) \exp(\xi^2/2\theta) \pm 1]$, respectively. For the classical particles the EDF is the Maxwell-Boltzmann (MB) distribution function, $f_{MB}^{(eq)} = \rho_0 \exp(-\xi^2/2\theta)/(2\pi\theta)^{D/2}$, where ρ_0 is a dimensionless density. All variables are defined dimensionless ($m = k_B = c = \hbar = 1$) by means of the scale set by T_r and its associated velocity, c_r .

The macroscopic observable quantities follow from the distribution function and to obtain the macroscopic hydrodynamical equations the so-called Chapman-Enskog assumption must be imposed¹⁴, which means that the first three moments are either computable from the non-equilibrium distribution function or from the known EDF¹⁵. The presence of a macroscopic velocity shifts the distribution of microscopic velocities such that the first three moments are given by, $\rho \equiv \int d^D \xi f^{(eq)}(\xi - \mathbf{u})$, $\rho \mathbf{u} \equiv \int d^D \xi \xi f^{(eq)}(\xi - \mathbf{u})$ and $D\rho\bar{\theta} \equiv \int d^D \xi (\xi - \mathbf{u})^2 f^{(eq)}(\xi - \mathbf{u})$, where the third moment is the internal energy, defined through the pseudo-temperature¹⁴, which in the classical case becomes the temperature, $\bar{\theta} = \theta$.

Generalized polynomials in D dimensions. – Consider the D-dimensional space of the microscopic velocity, $\xi \equiv (\xi_1, \xi_2, \dots, \xi_D)$, endowed with a weight function that can be, for instance, any one of the three EDF's previously discussed: $\omega = (f_{FD}^{(eq)}, f_{BE}^{(eq)}, f_{MB}^{(eq)})$. We claim here the existence of a set of orthonormal polynomials $\mathcal{P}_{i_1 \dots i_N}(\xi)$, such that,

$$\int d^D \xi \omega(\xi) \mathcal{P}_{i_1 \dots i_N}(\xi) \mathcal{P}_{j_1 \dots j_M}(\xi) = \delta_{NM} \delta_{i_1 \dots i_N | j_1 \dots j_M},$$

$$\omega(\xi) \equiv f^{(eq)}(\xi). \quad (1)$$

The D-dimensional Hermite polynomials correspond to the particular set associated to the Gaussian weight function, $\omega(\xi) = \exp(-\xi^2/2)/(2\pi)^{D/2}$. In fact the existence of the generalised polynomials only rely on some general properties of the weight function, such as dependence on $\xi \equiv |\xi|$ and its evanishment at extremely high microscopic velocities, $\omega(\xi) \rightarrow 0$ for $\xi \rightarrow \infty$, such that the integrals I_N given below, are well defined.

$$\int d^D \xi \omega(\xi) \xi_{i_1} \dots \xi_{i_N} = I_N \delta_{i_1 \dots i_N}, \quad (2)$$

By symmetry it holds that $I_N = 0$ for N odd. We introduce the following tensors which are sums of products of the Kronecker's delta function ($\delta^{ij} = 1$ for $i = j$ and 0 for $i \neq j$), namely, $\delta_{i_1 \dots i_N | j_1 \dots j_N} \equiv \delta_{i_1 j_1} \dots \delta_{i_N j_N} +$ all permutations of j 's and $\delta_{i_1 \dots i_N j_1 \dots j_N} \equiv \delta_{i_1 j_1} \dots \delta_{i_N j_N} +$ all permutations. Thus they contain $N!$ and $(2N)!$ terms,

respectively. Using the aforementioned symmetries the I_{2N} are given by,

$$I_{2N} = \frac{\pi^{\frac{D}{2}}}{2^{N-1} \Gamma(N + \frac{D}{2})} \int_0^\infty d\xi \omega(\xi) \xi^{2N+D-1}. \quad (3)$$

A direct consequence of Eqs.(1) and (2) is that the moments are given by $\rho = I_0$ and $\bar{\theta} = I_2/I_0$. The polynomials $\mathcal{P}_{i_1 \dots i_N}(\xi)$ are of N^{th} order in ξ , symmetrical in the indices $i_1 \dots i_N$, and display the parity property $\mathcal{P}_{i_1 \dots i_N}(-\xi_{i_1}, \dots, -\xi_{i_k}, \dots, -\xi_{i_N}) = (-1)^N \mathcal{P}_{i_1 \dots i_N}(\xi_{i_1}, \dots, \xi_{i_k}, \dots, \xi_{i_N})$. They are tensors in Euclidean space expressed in terms of the vector components ξ_i and of δ_{ij} , like the Hermite polynomials themselves. The first four ones are given by,

$$\mathcal{P}_0(\xi) = c_0, \quad (4)$$

$$\mathcal{P}_{i_1}(\xi) = c_1 \xi_{i_1}, \quad (5)$$

$$\mathcal{P}_{i_1 i_2}(\xi) = c_2 \xi_{i_1} \xi_{i_2} + (\bar{c}_2 \xi^2 + c'_2) \delta_{i_1 i_2}, \quad (6)$$

$$\mathcal{P}_{i_1 i_2 i_3}(\xi) = c_3 \xi_{i_1} \xi_{i_2} \xi_{i_3} + (\bar{c}_3 \xi^2 + c'_3) (\xi_{i_1} \delta_{i_2 i_3} + \xi_{i_2} \delta_{i_1 i_3} + \xi_{i_3} \delta_{i_1 i_2}), \quad \text{and}, \quad (7)$$

$$\mathcal{P}_{i_1 i_2 i_3 i_4}(\xi) = c_4 \xi_{i_1} \xi_{i_2} \xi_{i_3} \xi_{i_4} + (\bar{c}_4 \xi^2 + c'_4) (\xi_{i_1} \xi_{i_2} \delta_{i_3 i_4} + \xi_{i_1} \xi_{i_3} \delta_{i_2 i_4} + \xi_{i_1} \xi_{i_4} \delta_{i_2 i_3} + \xi_{i_2} \xi_{i_3} \delta_{i_1 i_4} + \xi_{i_2} \xi_{i_4} \delta_{i_1 i_3} + \xi_{i_3} \xi_{i_4} \delta_{i_1 i_2}) + (\bar{d}_4 \xi^4 + d'_4 \xi^2 + d_4) \delta_{i_1 i_2 i_3 i_4}. \quad (8)$$

All the coefficients are solely functions of the integrals I_{2N} : $c_K = 1/\sqrt{I_{2K}}$ for $K = 0, 1, 2, 3$, and 4, and $c'_K = -c_K (I_{2K-2}/I_{2K-4}) \Delta_{2K-2}$, $\bar{c}_K = c_K (-1 + \Delta_{2K-2})/(D + 2K - 4)$ for $K = 1, 2, 3$, and 4. We define for L even, $\Delta_L^2 \equiv 2/[(D + L) - J_L(D + L - 2)]$ and $J_L \equiv I_L^2/I_{L+2}I_{L-2}$. The remaining coefficients are $d_4^2 = (8\delta_4^2 I_4/\delta_2)/[\delta_2 \delta_6 (D + 4) - \delta_4^2 D]$, $d'_4 = -d_4 [I_0/I_2 + I_4 \delta_2/(I_2 \delta_4)]/D + 2c_4 I_6 \Delta_6/(DI_4)$ and $\bar{d}_4 = d_4 \delta_2/[D(D + 2)\delta_4] + c_4 [D - 2(D + 2)\Delta_6]/[D(D + 2)(D + 4)]$ where $\delta_L \equiv 2I_{L+2}I_{L-2}/\Delta_L^2$.

For the FD (+) and BE (-) weights one obtains that $I_{2N} = (2\pi)^{D/2} \theta^{N+D/2} g_{(N+D/2)}(e^{\mu/\theta})$, where $g_\nu(z) = \int_0^\infty dx x^{\nu-1}/(z^{-1}e^x \pm 1)$. The FD weight in the so-called Sommerfeld's limit¹⁶, used for the treatment of electrons in metals, becomes,

$$I_{2N} = \frac{(2\pi)^{\frac{D}{2}} \mu^{N+\frac{D}{2}}}{\Gamma(N + \frac{D}{2} + 1)} \left[1 + \frac{\pi^2}{6} \left(N + \frac{D}{2} \right) \left(N + \frac{D}{2} - 1 \right) \left(\frac{\theta}{\mu} \right)^2 + \dots \right], \quad (9)$$

where $\theta/\mu \ll 1$ such that terms of order $(\theta/\mu)^4$ and higher are disregarded. For the MB weight $I_{2N} = I_0 \theta^N$ and $\mathcal{P}_{i_1 i_2 \dots i_N}(\xi) = \mathcal{H}_{i_1 i_2 \dots i_N}(\xi/\sqrt{\theta})/\sqrt{I_0}$, where $\mathcal{H}_{i_1 i_2 \dots i_N}$ are the Hermite polynomials. In the Gaussian limit $I_{2N} = 1$, and so the coefficients proportional to ξ^2 and ξ^4 , that multiply the tensors $(\xi_{i_1} \xi_{i_1} \dots) \delta_{i_3 i_4 \dots}$, vanish. The remaining ones becomes equal to ± 1 : $c_K = 1$,

$|c'_K| = 1$, $\bar{c}_K = 0$, $|d_4| = 1$, $\bar{d}_4 = 0$, and $d'_4 = 0$.

The expansion in powers of the generalised polynomials. – Consider an EDF expanded in the standard way, where the coefficients are determined by the orthonormality of the polynomial basis,

$$f^{(eq)}(\xi - \mathbf{u}) = \omega(\xi) \sum_{N=0}^K \frac{1}{N!} \mathcal{A}_{i_1 i_2 \dots i_N}(\mathbf{u}) \mathcal{P}_{i_1 i_2 \dots i_N}(\xi) \quad (10)$$

$$\mathcal{A}_{i_1 i_2 \dots i_N}(\mathbf{u}) = \int d^D \xi f^{(eq)}(\xi - \mathbf{u}) \mathcal{P}_{i_1 i_2 \dots i_N}(\xi). \quad (11)$$

However because the EDF is the weight itself, define $\boldsymbol{\eta} \equiv \xi - \mathbf{u}$ to obtain that,

$$\mathcal{A}_{i_1 i_2 \dots i_N}(\mathbf{u}) = \int d^D \boldsymbol{\eta} \omega(\boldsymbol{\eta}) \mathcal{P}_{i_1 i_2 \dots i_N}(\boldsymbol{\eta} + \mathbf{u}), \quad (12)$$

It is advantageous to expand the polynomial $\mathcal{P}_{i_1 i_2 \dots i_N}(\boldsymbol{\eta} + \mathbf{u})$ in a sum over polynomials $\mathcal{P}_{i_1 i_2 \dots i_M}(\boldsymbol{\eta})$ of equal or lower order ($M \leq N$). This is because, according to Eq.(1), only the term proportional to $\mathcal{P}_0(\boldsymbol{\eta})$ matters, since only the lowest order term is a constant. Therefore from $\mathcal{P}_{i_1 i_2 \dots i_N}(\boldsymbol{\eta} + \mathbf{u}) = \dots + \mathcal{P}'_{i_1 i_2 \dots i_N}(\mathbf{u}) \mathcal{P}_0(\boldsymbol{\eta})/c_0$, where \mathcal{P}' is a polynomial of the same order of \mathcal{P} but with distinct coefficients, it is straightforward to obtain that $\mathcal{A}_{i_1 i_2 \dots i_N}(\mathbf{u}) = \mathcal{P}'_{i_1 i_2 \dots i_N}(\mathbf{u}) I_0$. The first four coefficients directly obtained from Eq.(12) are given by,

$$\mathcal{A}_0(\mathbf{u}) = I_0 c_0, \quad (13)$$

$$\mathcal{A}_{i_1}(\mathbf{u}) = I_0 c_1 u_{i_1}, \quad (14)$$

$$\mathcal{A}_{i_1 i_2}(\mathbf{u}) = I_0 (c_2 u_{i_1} u_{i_2} + \bar{c}_2 \mathbf{u}^2 \delta_{i_1 i_2}), \quad (15)$$

$$\mathcal{A}_{i_1 i_2 i_3}(\mathbf{u}) = I_0 \{c_3 u_{i_1} u_{i_2} u_{i_3} + [c'_3 (1 - J_2) + \bar{c}_3 \mathbf{u}^2] \cdot (u_{i_1} \delta_{i_2 i_3} + u_{i_2} \delta_{i_1 i_3} + u_{i_3} \delta_{i_1 i_2})\}, \quad \text{and}, \quad (16)$$

$$\mathcal{A}_{i_1 i_2 i_3 i_4}(\mathbf{u}) = I_0 \left\{ c_4 u_{i_1} u_{i_2} u_{i_3} u_{i_4} + [(1 - J_2 J_4) c'_4 + \bar{c}_4 \mathbf{u}^2] \cdot (u_{i_1} u_{i_2} \delta_{i_3 i_4} + u_{i_1} u_{i_3} \delta_{i_2 i_4} + u_{i_1} u_{i_4} \delta_{i_2 i_3} + u_{i_2} u_{i_3} \delta_{i_1 i_4} + u_{i_2} u_{i_4} \delta_{i_1 i_3} + u_{i_3} u_{i_4} \delta_{i_1 i_2}) + \left[\left(2 \frac{I_2}{I_0} (\bar{c}_4 + (D+2)\bar{d}_4) + d'_4 \right) \mathbf{u}^2 + \bar{d}_4 \mathbf{u}^4 \right] \delta_{i_1 i_2 i_3 i_4} \right\}. \quad (17)$$

It can be proven that $\mathcal{A}_{i_1 i_2 \dots i_N}(\mathbf{u} = 0) = 0$ for $N > 0$ by taking $\mathbf{u} = 0$ in Eq.(10).

$$\sum_{N=0}^K \frac{1}{N!} \mathcal{A}_{i_1 i_2 \dots i_N}(\mathbf{u} = 0) \mathcal{P}_{i_1 i_2 \dots i_N}(\xi) = 1. \quad (18)$$

Since $\mathcal{A}_0 \mathcal{P}_0 = 1$ from Eqs.(4) and (13), all the remaining coefficients must vanish.

The decomposition of the EDF in terms of the new set of polynomials displays notorious advantages over any another choice of weight. The displacement velocity controls the smallness of the coefficients $\mathcal{A}_{i_1 i_2 \dots i_N}$, which to the lowest order are linear and quadratic in \mathbf{u} for the odd and even ($N > 0$) coefficients, respectively. Recall that a distinct choice of weight can render coef-

ficients that do not necessarily vanish for $\mathbf{u} = 0$, and so, have other parameters that control their smallness. This is the case of the thermal LBM developed over the gaussian weight, whose coefficients also depends on the temperature deviation^{14,17} from a reference value, $\theta - 1$. In this letter we show that the use of the generalised polynomials turns the ratio u/c_0 the expanding parameter for the EDF in replacement of u/c_r . This property was only found in specific examples, such as for the isothermal LBM⁵, relativistic systems¹⁰ and graphene¹¹. Recall that both \mathbf{u} and ξ are so far normalized by c_r . A glimpse into this change of scale is provided by the highest order term in the velocities of the polynomials, $\mathcal{P}_{i_1 i_2 \dots i_N}(\xi) = (1/\sqrt{I_{2N}}) \xi_{i_1} \xi_{i_2} \dots \xi_{i_N} + \dots$, and the coefficients, $\mathcal{A}_{i_1 i_2 \dots i_N}(\mathbf{u}) = (I_0/\sqrt{I_{2N}}) u_{i_1} u_{i_2} \dots u_{i_N} + \dots$. The presence of the I_{2N} introduces the scale $c_0 = (I_{2N})^{1/2N}$, and we define $\xi' \equiv \xi/c_0$ and $\mathbf{u}' \equiv \mathbf{u}/c_0$ such that $\mathcal{P}_{i_1 i_2 \dots i_N}(\xi) = \xi'_{i_1} \xi'_{i_2} \dots \xi'_{i_N} + \dots$, and, $\mathcal{A}_{i_1 i_2 \dots i_N}(\mathbf{u}) = I_0 u'_{i_1} u'_{i_2} \dots u'_{i_N} + \dots$. Consequently c_r is no longer present, as can be checked, for instance, in the Sommerfeld limit of the FD weight, Eq.(9), since $c_0 \sim \sqrt{\mu}$ and because μ is also normalized by c_r^2 . The overall result is that the chemical potential sets the new velocity scale. Recall that to achieve conservation of mass and momentum it is enough to carry the expansion of Eq.(10) to order $N = 3$, and to include energy conservation one order further, $N = 4$, and no more¹⁴. Thus we find that convergence is achieved in such orders, just demanding a small ratio u/c_0 .

The Gaussian quadrature. – The Gaussian quadrature provides a way to calculate the integral of a function as a sum over some of its values at a set of points α multiplied by pre-determined weights w_α . We claim here that the Gaussian quadrature holds for $\omega(\xi) = f^{(eq)}(\xi)$, such that $\int d^D \xi \omega(\xi) g(\xi) = \sum_\alpha w_\alpha g_\alpha$, $g_\alpha \equiv g(\xi_\alpha)$. Thus Eqs.(3) turn into the following relations.

$$\sum_\alpha w_\alpha \xi_{\alpha i_1} \dots \xi_{\alpha i_N} = I_N \delta_{i_1 \dots i_N}, \quad (19)$$

such that $I_N = 0$ for N odd. The passage from the continuous to the discrete consists in determining lattices, namely the sets of $\xi_{\alpha i_1}$ and w_α that satisfy Eqs.(19) once known the integrals I_{2N} of Eqs.(2). In the present approach the weights becomes function of the above integrals, $w_\alpha(I_{2N})$, and must be updated at each time step of the evolution procedure.

The normalization of the velocities changes from c_r to c_0 once the lattice of *geometrical velocities*, defined by the set of α vectors $\mathbf{e}_\alpha \equiv \xi_\alpha/c_0$, is introduced, to solve Eqs.(19). The \mathbf{e}_α form a basis that generate an array of regularly spaced points in real space while the lattice of vectors ξ_α does not, and so, must be locally adjusted by c_0 . Hereafter we restrict our study to the $N = 3$ order in the EDF of Eq.(10) which renders the Navier-Stokes equation from the Chapman-Enskog analysis of the Boltzmann-BGK equation¹⁴. Therefore, as examples, we consider the following two lattices. The

$d1v3$ ($D = 1$) lattice contains only three velocities, $\mathbf{e}_0 = 0$, $\mathbf{e}_{\pm 1} = \pm 1$ and the associated weights are $w_0 = I_0(1 - J_2/3)$ and $w_{\pm 1} = I_0 J_2/6$. The lattice $d2v9$ ($D = 2$) has its geometrical velocities on the sides and diagonals of a square, namely, \mathbf{e}_α , $\alpha = 0, \dots, 8$: $\mathbf{e}_0 = (0, 0)$, $\mathbf{e}_{long} = [(1, 1), (1, -1), (-1, -1), (-1, 1)]$ and $\mathbf{e}_{short} = [(1, 0), (0, 1), (-1, 0), (0, -1)]$. The corresponding weights are $w_0 = I_0(1 - J_2/3)$, $w_l = I_0 J_2/36$ and $w_s = I_0 J_2/9$, associated to the center, long and short vectors, respectively. For both lattices $c_0 = \sqrt{3I_4/I_2}$. Therefore the $N = 3$ EDF,

$$f_\alpha^{(eq)} = w_\alpha \left\{ 1 + \frac{I_0}{I_2} \boldsymbol{\xi}_\alpha \cdot \mathbf{u} + \frac{I_0}{2I_4} (\boldsymbol{\xi}_\alpha \cdot \mathbf{u})^2 - \frac{I_0 (\Delta_2^2 - 1)}{2I_4 D} \mathbf{u}^2 \boldsymbol{\xi}_\alpha^2 - \frac{I_2}{2I_4} \Delta_2^2 \mathbf{u}^2 + \frac{1}{6} I_0 (\boldsymbol{\xi}_\alpha \cdot \mathbf{u}) \cdot \left[3(1 - J_2) \frac{J_4}{I_2} (D + 2) \Delta_4^2 - 3(1 - J_2) \frac{J_4}{I_4} \Delta_4^2 \boldsymbol{\xi}_\alpha^2 - 3 \frac{J_4}{I_4} \Delta_4^2 \mathbf{u}^2 + 3 \frac{\Delta_4^2 - 1}{I_6 (D + 2)} \boldsymbol{\xi}_\alpha^2 \mathbf{u}^2 + \frac{1}{I_6} (\boldsymbol{\xi}_\alpha \cdot \mathbf{u})^2 \right] \right\},$$

becomes in the geometrical space,

$$f_\alpha^{(eq)} = w_\alpha \left\{ 1 + \frac{3}{J_2} \mathbf{e}_\alpha \cdot \mathbf{u}_0 + \frac{9}{2J_2} (\mathbf{e}_\alpha \cdot \mathbf{u}_0)^2 + \frac{9}{2} \left(\frac{\Delta_2^2 - 1}{J_2 D} \right) \mathbf{u}_0^2 \mathbf{e}_\alpha^2 - \frac{3}{2} \Delta_2^2 \mathbf{u}_0^2 + \frac{3}{2J_2} (\mathbf{e}_\alpha \cdot \mathbf{u}_0) \cdot \left[(1 - J_2) J_4 (D + 2) \Delta_4^2 - 3(1 - J_2) J_4 \Delta_4^2 \mathbf{e}_\alpha^2 - 3J_4 \Delta_4^2 \mathbf{u}_0^2 + 9 \frac{J_4 (\Delta_4^2 - 1)}{D + 2} \mathbf{u}_0^2 \mathbf{e}_\alpha^2 + 3J_4 (\mathbf{e}_\alpha \cdot \mathbf{u}_0)^2 \right] \right\} \quad (20)$$

Updates are done in the density and in the normalized velocity,

$$I_0 = \sum_\alpha f_\alpha^{(eq)} \mathbf{e}_\alpha \quad \text{and} \quad \mathbf{u}_0 \equiv \frac{\mathbf{u}}{c_0} = \frac{1}{I_0} \sum_\alpha f_\alpha^{(eq)} \mathbf{e}_\alpha. \quad (21)$$

Then the lattice Boltzmann-BGK equation becomes¹⁸, $f_\alpha(\mathbf{x} + \mathbf{e}_\alpha, t + 1) - f_\alpha(\mathbf{x}, t) = -[f_\alpha(\mathbf{x}, t) - f_\alpha^{(eq)}(\mathbf{x}, t)]/\tau$. In conclusion c_0 has become the scale velocity in replace-

ment of c_τ for the two lattices considered above.

The Riemann problem for zero temperature fermions. – The one-dimensional Navier-Stokes equation is independent of τ because the viscosity stress tensor vanishes for $D = 1$, and so reduces to the Euler equation¹⁴. A direct comparison between a known solution of the Euler equation in case of a stepwise initial condition, the so-called Riemann problem¹⁹, and our numerical solution of the one-dimensional Boltzmann equation is possible. We take fermions at zero temperature, that is, $\theta = 0$ at Eq.(9). This renders $I_0 = (2\pi\mu)^{D/2}/\Gamma(D/2 + 1)$ and $J_2 = (D + 4)/(D + 2)$. Local changes in ρ and u are only possible due to local variations in the chemical potential. We use the two lattices, $d1v3$ ($w_0 = 7I_0/12$ and $w_{\pm 1} = 5I_0/24$) and $d2v9$ ($w_0 = 3I_0/5$, $w_l = I_0/30$ and $w_s = 2I_0/15$). In this case the weights are constant in all lattice points with a remaining multiplicative density played by I_0 such as in the known LBM method⁵. The velocity scale, $c_0 = \sqrt{6\mu/(D + 4)}$, is essentially the Fermi velocity. For both lattices we find independence of the numerical solution with respect to τ within a wide range ($\tau = 0.57$ - 0.95 , reduced units). Fig. 1 shows the comparison between the obtained ρ and u with the known Riemann problem. Fermions at zero temperature obey an isothermal perfect gas law where the temperature is the Fermi temperature itself. Fig. 1 shows this comparison between the shock wave solution for an isothermal gas and the LBM simulations for the lattices $d1v3$ and $d2v9$ after 500 steps. We find good agreement for the density and velocity curves of the $d1v3$ and $d2v9$ lattices. Comparison between the classical and quantum gas velocities shows the same density and reveals distinct results for the velocity because of the normalization.

Conclusion - We propose orthonormal polynomials that generalize the D-dimensional Hermite polynomials and obtain a convergent LBM for a small ratio between the displacement velocity and the true scale velocity.

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